A Generalization of Macaulay's Method with Applications in Structural Mechanics

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Some recent references to the use of Macaulay brackets in the analysis of the deflections of beams under discontinuous lateral loads suggest that it is not widely realized that the concept can easily be generalized to apply to a wide range of problems, particularly those occurring in structural mechanics. In this paper the logical basis of the method is explained, and detailed expressions are derived in a form suitable for immediate application to a wide range of static and pulsating beam problems and to some axially symmetrical plate problems.

Introduction

TWO recent notes[1, 2] that draw attention to the use of Macaulay[3] brackets in the analysis of the deflections of beams, beam columns, and tie rods under discontinuous loading, together with a subsequent review[4] of Ref. 2, strongly suggest that their use, even in these simple cases, is not widely known. In a subsequent note,[5] which is primarily a historical review, Weissenburger attributes the method to Clebsch,[6] who, like Macaulay, was concerned with the deflections of simple beams. Weissenburger also hints that there are "possible simplifications to be achieved in other areas by judicious application of the method."

This has, in fact, been known for many years, and the method was used by Case, as early as 1932 in the second edition of his textbook,[7] for the solution of problems concerned with the steady vibration of beams under discontinuous pulsating loads. Judging by the preface to this second edition, it seems almost certain that the same material was also included in the first edition, published in 1925, which is unfortunately unavailable to the author. Moreover, all of the forementioned applications have been taught to engineering undergraduates at Cambridge University for at least 30 years and in all probability for 40 years. The possibility of extension to other beam problems, such as the case of a beam supported on an elastic foundation, was certainly known to H. A. Webb and taught to at least one of his former students almost 25 years ago.

It was, therefore, with considerable diffidence that this paper was written, since the author feels that its contents must be known in principle to many others. Nevertheless, the recently published notes appear to demand a more complete summary of the method than has hitherto appeared, and the paper is written with this in view.

Despite the apparent priority of Clebsch's work, the technique will be referred to here as Macaulay's method in order to conform with the widely accepted usage. The logical basis of the method is explained in its most general terms. Subsequently, detailed expressions for the terms that appear within Macaulay brackets in equations for the deflections of beams and axially symmetrical plates are presented in a form suitable for immediate application. The beam problems cover all possible combinations of axial load and continuous elastic support, for each of four types of discontinuous lateral load, and also the case of pulsating lateral loads. Some of these beam problems, and all of the plate problems, have not been treated previously by Macaulay's method in the published literature, as far as the author is aware.

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Definitions

1) If $f(x)$ is some function of the independent variable x, we define the "Macaulay bracket" $\langle\ \rangle_a$ as follows:

$$\langle f(x)\rangle_a = 0 \text{ if } x < a$$
$$= f(x) \text{ if } x \geqq a$$

2) Define the symbol $\Delta f(a)$ as meaning the discontinuity in a function $f(x)$ as x passes through the point $x = a$. Thus,

$$\Delta f(a) = f(a + 0) - f(a - 0)$$

Generalization of Macaulay's Method

Macaulay's method can be generalized to apply to any problem of the following type.

1) The problem is governed by an ordinary differential equation of the form

$$L(y) = R(x) \tag{1}$$

where $L(\ )$ represents a linear differential operator of any order, n say, in which the coefficients of the various derivatives, though not necessarily constant, are continuous throughout the range $0 \leqq x \leqq l$.

2) The right-hand side $R(x)$ and its various derivatives may be discontinuous at a series of points $x = a_i (i = 1, 2, \ldots, N)$ within the range $0 < x < l$ and can be expressed in the form

$$R(x) = R_0(x) + \sum_{i=1}^{N} \langle R_i(x)\rangle_{a_i} \tag{2}$$

where each of the functions R_0 and R_i is continuous.

3) The required general solution $y(x)$ and its first $(n - 1)$ derivatives satisfy specified conditions of continuity or discontinuity at each of the N points $x = a_i$. Thus, the values of

$$\Delta y(a_i) \qquad \Delta y'(a_i) \qquad \Delta y''(a_i), \ldots, \qquad \Delta y^{n-1}(a_i) \qquad i = 1, 2, \ldots N$$

are specified. (This general solution will contain n arbitrary constants of integration which will be determined from the prescribed conditions at the ends of the range, $x = 0$ and $x = l$.)

Let the general solution of the nth-order homogeneous differential equation $L(y) = 0$ be

$$y_H(x) = \sum_{r=1}^{n} A_r\, u_r(x) \tag{3}$$

where the n constants of integration A_r are arbitrary. Then the required solution of Eq. (1), which satisfies the continuity

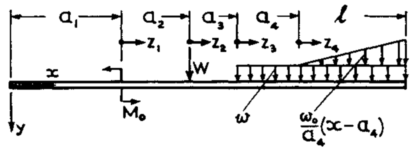

Fig. 1 Types of discontinuous loading on a beam.

conditions at the N points $x = a_i$, can be written in the form

$$y(x) = y_H(x) + I_0(x) + \sum_{i=1}^{N} \langle G_i(x) \rangle_{a_i} \tag{4}$$

where $I_0(x)$ and the $G_i(x)$ are continuous functions. The function $I_0(x)$ is *any* particular integral of the equation

$$L(y) = R_0(x)$$

whereas the functions $G_i(x)$ are particular integrals of the equations

$$L(y) = R_i(x) \qquad i = 1, 2 \ldots N \tag{5}$$

Moreover, since any discontinuities in $y(x)$ or its various derivatives that occur in Eq. (4) are entirely due to the Macaulay brackets, it is seen that

$$\begin{aligned} G_i(a_i) &= \Delta y(a_i) \\ G_i'(a_i) &= \Delta y'(a_i) \\ &\;\vdots \\ G_i^{n-1}(a_i) &= \Delta y^{n-1}(a_i) \qquad (i = 1, 2 \ldots N) \end{aligned} \tag{6}$$

Thus the problem is reduced to determining particular integrals $G_i(x)$ of Eq. (5) which satisfy Eqs. (6). Suppose that one particular integral $I_i(x)$ of Eq. (5) is known. In general, this will not satisfy Eqs. (6), but the difference between $I_i(x)$ and $G_i(x)$ must be a solution of the homogeneous equation $L(y) = 0$. Hence,

$$G_i(x) = I_i(x) + \sum_{r=1}^{n} B_{ri}\, u_r(x) \tag{7}$$

Using Eqs. (6), the n constants B_{ri} can be obtained by solving the following n simultaneous equations:

$$\begin{aligned} \sum_{r=1}^{n} B_{ri}\, u_r(a_i) &= \Delta y(a_i) - I_i(a_i) \\ \sum_{r=1}^{n} B_{ri}\, u_r'(a_i) &= \Delta y'(a_i) - I_i'(a_i) \\ &\;\vdots \\ \sum_{r=1}^{n} B_{ri}\, u_r^{n-1}(a_i) &= \Delta y^{n-1}(a_i) - I_i^{n-1}(a_i) \end{aligned} \tag{8}$$

The n constants A_r which occur in the expression for $y_H(x)$ in Eq. (4) can be determined only from the prescribed conditions at $x = 0$ and $x = l$.

It should be noted that, if the general solution $y_H(x)$ of the homogeneous differential equation is known, it is always possible to find one particular integral $I_i(x)$ of Eq. (5) using the method of variation of parameters.[8]

In most structural problems to which Macaulay's method is applicable, the variable y will represent the deflection at a station defined by the value of x, and geometrical continuity conditions will invariably require that y and y' should be continuous over the whole range of x. The discontinuities in $R(x)$ arise because of discontinuities in loading, and the discontinuities in y'' and y''' are easily related to discontinuities in bending moment and shear force.

It should be emphasized that in any problem the terms $G_i(x)$ corresponding to given types of load discontinuity are independent of the boundary conditions at $x = 0$ and $x = l$ and can be determined for all time. Once they have been cataloged, the solution for a problem with *any* number N of load discontinuities only requires the determination of the n constants of integration A_r in the expression for $y_H(x)$.

Some Applications to Beam Problems

Problem 1. A Uniform Beam, Supported on an Elastic Foundation, and Subjected to Axial Compression and Discontinuous Lateral Loads

Consider a uniform beam of length l and flexural rigidity EI that is supported on an elastic foundation of constant stiffness K, so that, if a deflection $y(x)$ occurs, there is a restoring force of magnitude $Ky(x)$ per unit length. Suppose that the beam is subjected to an axial compressive force P and also to lateral loads. The governing differential equation is

$$EIy'''' + Py'' + Ky = p(x) \tag{9}$$

where $p(x)$ is the applied lateral load per unit length.

Consider the effect of the four types of discontinuous load shown in Fig. 1, namely a concentrated bending moment M_0 at $x = a_1$, a concentrated lateral load W at $x = a_2$, a uniformly distributed load w per unit length over the range $a_3 \leqq x \leqq l$, and a linearly varying load of intensity $(w_0/a_4)(x - a_4)$ per unit length over the range $a_4 \leqq x \leqq l$.

Because of these loads alone, the right-hand side of Eq. (9) can be written as

$$p(x) = w\langle 1 \rangle_{a_3} + (w_0/a_4)\langle x - a_4 \rangle_{a_4} \tag{10}$$

Since Eq. (9) is of fourth-order, we require four continuity conditions at each of the points $x = a_1$, a_2, a_3 and a_4. These are as follows:

$$\begin{aligned} \Delta y(a_i) &= \Delta y'(a_i) = 0 \qquad (i = 1, 2, 3, 4) \\ \Delta y''(a_1) &= M_0/EI \qquad \Delta y'''(a_1) = 0 \\ \Delta y''(a_2) &= 0 \qquad \Delta y'''(a_2) = W/EI \\ \Delta y''(a_3) &= \Delta y'''(a_3) = \Delta y''(a_4) = \Delta y'''(a_4) = 0 \end{aligned} \tag{11}$$

Corresponding to each of the four loads there is a function $G_i(x)$ that appears within Macaulay brackets in Eq. (4). Expressions for the solution y_H of the homogeneous differential equation and for the four terms $G_i(x)$ are listed below for each of three cases, depending upon whether P is greater than, equal to, or less than $2(EIK)^{1/2}$. In these expressions the symbols α, μ, λ, θ, ξ, and η are defined as follows:

$$\begin{aligned} \alpha &= (P/EI)^{1/2} \qquad \mu = (K/4EI)^{1/4} \\ \lambda &= \mu 2^{1/2} = (K/EI)^{1/4} \\ \theta &= (\mu^2 + \tfrac{1}{4}\alpha^2)^{1/2} \qquad \xi = (\tfrac{1}{4}\alpha^2 - \mu^2)^{1/2} \\ \eta &= i\xi = (\mu^2 - \tfrac{1}{4}\alpha^2)^{1/2} \end{aligned} \tag{12}$$

Also, for conciseness, the symbols z_i have been introduced, where

$$z_i = x - a_i \qquad i = 1, 2, 3, 4$$

Thus, z_i is the distance coordinate measured along the axis of the beam from an origin at the point $x = a_i$, as indicated in Fig. 1.

Case a) $P > 2(EIK)^{1/2}$, i.e., $\alpha^2 > 4\mu^2$

$$y_H(x) = (A_1 \cos\xi x + A_2 \sin\xi x)\cos\theta x + (A_3 \cos\xi x + A_4 \sin\xi x)\sin\theta x \tag{13}$$

$$\left.\begin{aligned} G_1(x) &= (M_0/2EI\theta\xi)\sin\theta z_1 \sin\xi z_1 \\ G_2(x) &= (W/4EI\mu^2)(\theta^{-1}\sin\theta z_2 \cos\xi z_2 - \xi^{-1}\cos\theta z_2 \sin\xi z_2) \\ G_3(x) &= (w/K)[1 - \cos\theta z_3 \cos\xi z_3 - (\alpha^2/4\theta\xi)\sin\theta z_3 \sin\xi z_3] \\ G_4(x) &= (w_0/Ka_4)[z_4 - \tfrac{1}{2}\{1 + (\alpha^2/2\mu^2)\}\theta^{-1}\sin\theta z_4 \cos\xi z_4 - \tfrac{1}{2}\{1 - (\alpha^2/2\mu^2)\}\xi^{-1}\cos\theta z_4 \sin\xi z_4] \end{aligned}\right\} \tag{14}$$

Case b) $P = 2(EIK)^{1/2}$, i.e., $\alpha^2 = 4\mu^2$

$$y_H(x) = (A_1 + A_2 x)\cos\lambda x + (A_3 + A_4 x)\sin\lambda x \quad (15)$$

$$\left.\begin{aligned} G_1(x) &= (M_0/2EI\lambda) z_1 \sin\lambda z_1 \\ G_2(x) &= (W/2EI\lambda^3)(\sin\lambda z_2 - \lambda z_2 \cos\lambda z_2) \\ G_3(x) &= (w/K)(1 - \cos\lambda z_3 - \tfrac{1}{2}\lambda z_3 \sin\lambda z_3) \\ G_4(x) &= (w_0/Ka_4)[z_4 - (3/2\lambda)\sin\lambda z_4 + \tfrac{1}{2} z_4 \cos\lambda z_4] \end{aligned}\right\} \quad (16)$$

Case c) $P < 2(EIK)^{1/2}$, i.e., $\alpha^2 < 4\mu^2$

$$y_H(x) = (A_1 \cosh\eta x + A_2 \sinh\eta x)\cos\theta x + (A_3 \cosh\eta x + A_4 \sinh\eta x)\sin\theta x \quad (17)$$

$$\left.\begin{aligned} G_1(x) &= (M_0/2EI\theta\eta)\sin\theta z_1 \sinh\eta z_1 \\ G_2(x) &= (W/4EI\mu^2)(\theta^{-1}\sin\theta z_2 \cosh\eta z_2 - \eta^{-1}\cos\theta z_2 \sinh\eta z_2) \\ G_3(x) &= (w/K)[1 - \cos\theta z_3 \cosh\eta z_3 - (\alpha^2/4\theta\eta)\sin\theta z_3 \sinh\eta z_3] \\ G_4(x) &= (w_0/Ka_4)[z_4 - \tfrac{1}{2}\{1 + (\alpha^2/2\mu^2)\}\theta^{-1}\sin\theta z_4 \cosh\eta z_4 - \tfrac{1}{2}\{1 - (\alpha^2/2\mu^2)\}\eta^{-1}\cos\theta z_4 \sinh\eta z_4] \end{aligned}\right\} \quad (18)$$

It should be mentioned that the differential equation governing the radial displacement of the wall of a thin cylindrical tube compressed axially and subjected, in addition, to axially symmetrical radial loading is formally identical with Eq. (9), except that EI must be replaced by D, the flexural rigidity of the wall, and K by Et/r^2, where t is the thickness and r the radius. Also P now represents the axial compression per unit of circumference, and $p(x)$ is the radial load per unit area. With these changes, Eqs. (13–18) are valid for this problem. Similarly, Problems 2 and 3 that follow have obvious analogies in thin cylindrical tubes.

Problem 2. A Uniform Beam, Supported on an Elastic Foundation, and Subjected to Axial Tension and Discontinuous Lateral Loads

This is identical with Problem 1 except that the beam is now subjected to an axial tension T instead of the compression P. Let

$$\begin{gathered} \beta = (T/EI)^{1/2} \qquad \mu = (K/4EI)^{1/4} \\ \lambda = \mu 2^{1/2} = (K/EI)^{1/4} \\ \phi = (\mu^2 + \tfrac{1}{4}\beta^2)^{1/2} \qquad \rho = (\tfrac{1}{4}\beta^2 - \mu^2)^{1/2} \\ \sigma = i\rho = (\mu^2 - \tfrac{1}{4}\beta^2)^{1/2} \end{gathered} \quad (19)$$

The results are as follows.

Case a) $T > 2(EIK)^{1/2}$, i.e., $\beta^2 > 4\mu^2$

$$y_H(x) = (A_1 \cosh\rho x + A_2 \sinh\rho x)\cosh\phi x + (A_3 \cosh\rho x + A_4 \sinh\rho x)\sinh\phi x \quad (20)$$

$$\left.\begin{aligned} G_1(x) &= (M_0/2EI\phi\rho)\sinh\phi z_1 \sinh\rho z_1 \\ G_2(x) &= (W/4EI\mu^2)(\rho^{-1}\sinh\rho z_2 \cosh\phi z_2 - \phi^{-1}\sinh\phi z_2 \cosh\rho z_2) \\ G_3(x) &= (w/K)[1 - \cosh\phi z_3 \cosh\rho z_3 + (\beta^2/4\phi\rho)\sinh\phi z_3 \sinh\rho z_3] \\ G_4(x) &= (w_0/Ka_4)[z_4 - \tfrac{1}{2}\{1 + (\beta^2/2\mu^2)\}\phi^{-1} \times \sinh\phi z_4 \cosh\rho z_4 - \tfrac{1}{2}\{1 - (\beta^2/2\mu^2)\}\rho^{-1} \times \cosh\phi z_4 \sinh\rho z_4] \end{aligned}\right\} \quad (21)$$

Case b) $T = 2(EIK)^{1/2}$, i.e., $\beta^2 = 4\mu^2$

$$y_H(x) = (A_1 + A_2 x)\cosh\lambda x + (A_3 + A_4 x)\sinh\lambda x \quad (22)$$

$$\left.\begin{aligned} G_1(x) &= (M_0/2EI\lambda) z_1 \sinh\lambda z_1 \\ G_2(x) &= (W/2EI\lambda^3)(\lambda z_2 \cosh\lambda z_2 - \sinh\lambda z_2) \\ G_3(x) &= (w/K)(1 - \cosh\lambda z_3 + \tfrac{1}{2}\lambda z_3 \sinh\lambda z_3) \\ G_4(x) &= (w_0/Ka_4)[z_4 - (3/2\lambda)\sinh\lambda z_4 + \tfrac{1}{2} z_4 \cosh\lambda z_4] \end{aligned}\right\} \quad (23)$$

Case c) $T < 2(EIK)^{1/2}$, i.e., $\beta^2 < 4\mu^2$

$$y_H(x) = (A_1 \cos\sigma x + A_2 \sin\sigma x)\cosh\phi x + (A_3 \cos\sigma x + A_4 \sin\sigma x)\sinh\phi x \quad (24)$$

$$\left.\begin{aligned} G_1(x) &= (M_0/2EI\phi\sigma)\sinh\phi z_1 \sin\sigma z_1 \\ G_2(x) &= (W/4EI\mu^2)(\sigma^{-1}\sin\sigma z_2 \cosh\phi z_2 - \phi^{-1}\sinh\phi z_2 \cos\sigma z_2) \\ G_3(x) &= (w/K)[1 - \cosh\phi z_3 \cos\sigma z_3 + (\beta^2/4\phi\sigma)\sinh\phi z_3 \sin\sigma z_3] \\ G_4(x) &= (w_0/Ka_4)[z_4 - \tfrac{1}{2}\{1 + (\beta^2/2\mu^2)\}\phi^{-1} \times \sinh\phi z_4 \cos\sigma z_4 - \tfrac{1}{2}\{1 - (\beta^2/2\mu^2)\}\sigma^{-1}\cosh\phi z_4 \sin\sigma z_4] \end{aligned}\right\} \quad (25)$$

Problem 3. A Uniform Beam, Supported on an Elastic Foundation, Subjected to Discontinuous Lateral Loads Only

If the axial load is zero, the appropriate results can be obtained from Eqs. (17) and (18) by putting $\alpha = 0$ and $\theta = \eta = \mu$, or, from Eqs. (24) and (25), by putting $\beta = 0$ and $\phi = \sigma = \mu$. Thus,

$$y_H(x) = (A_1 \cosh\mu x + A_2 \sinh\mu x)\cos\mu x + (A_3 \cosh\mu x + A_4 \sinh\mu x)\sin\mu x \quad (26)$$

$$\left.\begin{aligned} G_1(x) &= (M_0/2EI\mu^2)\sin\mu z_1 \sinh\mu z_1 \\ G_2(x) &= (W/4EI\mu^3)(\sin\mu z_2 \cosh\mu z_2 - \cos\mu z_2 \sinh\mu z_2) \\ G_3(x) &= (w/K)(1 - \cos\mu z_3 \cosh\mu z_3) \\ G_4(x) &= (w_0/Ka_4)[z_4 - (1/2\mu)(\sin\mu z_4 \cosh\mu z_4 + \cos\mu z_4 \sinh\mu z_4)] \end{aligned}\right\} \quad (27)$$

Problem 4. A Uniform Beam Subjected to Axial Compression and Discontinuous Lateral Loads

The results for this case can be derived from Eqs. (14) by the limiting process $K \to 0$, $\mu \to 0$, $\theta \to \xi \to \alpha/2$. A more direct approach is to start from the differential equation

$$EIy'' + Py = M(x) \quad (28)$$

where $M(x)$ is the bending moment due to the lateral loads. As a result of the four discontinuous loads shown in Fig. 1 we have

$$M(x) = M_0\langle 1\rangle_{a_1} + W\langle x - a_2\rangle_{a_2} + (w/2)\langle x - a_3\rangle_{a_3}^2 + (w_0/6a_4)\langle x - a_4\rangle_{a_4}^3 \quad (29)$$

Since Eq. (28) is of second order, it is only necessary to specify that y and y' shall be continuous at $x = a_1, a_2, a_3, a_4$. The appropriate solution is as follows, with α defined as in Eqs. (12),

$$y_H(x) = A_1 \cos\alpha x + A_2 \sin\alpha x \quad (30)$$

$$\left.\begin{aligned} G_1(x) &= (M_0/P)(1 - \cos\alpha z_1) \\ G_2(x) &= (W/\alpha P)(\alpha z_2 - \sin\alpha z_2) \\ G_3(x) &= (w/2\alpha^2 P)(\alpha^2 z_3^2 - 2 + 2\cos\alpha z_3) \\ G_4(x) &= (w_0/6\alpha^3 P a_4)(\alpha^3 z_4^3 - 6\alpha z_4 + 6\sin\alpha z_4) \end{aligned}\right\} \quad (31)$$

The first three of Eqs. (31) agree with the corresponding expressions quoted by Urry.[2]

Problem 5. A Uniform Beam Subjected to Axial Tension and Discontinuous Lateral Loads

This is identical with Problem 4 except that the beam is subjected to an axial tension T instead of the compression P. The solution can be obtained by replacing P by $-T$ and α by $i\beta$ in Eqs. (30) and (31) where β is defined in Eqs. (19). Thus,

$$y_H(x) = A_1 \cosh\beta x + A_2 \sinh\beta x \quad (32)$$

$$
\begin{aligned}
G_1(x) &= (M_0/T)(\cosh\beta z_1 - 1) \\
G_2(x) &= (W/\beta T)(\sinh\beta z_2 - \beta z_2) \\
G_3(x) &= (w/2\beta^2 T)(2\cosh\beta z_3 - 2 - \beta^2 z_3^2) \\
G_4(x) &= (w_0/6\beta^3 T a_4)(6\sinh\beta z_4 - 6\beta z_4 - \beta^3 z_4^3)
\end{aligned}
\tag{33}
$$

Problem 6. A Uniform Beam Subjected to Discontinuous Lateral Loads Only

The case of a simple beam, as originally considered by Macaulay, can be obtained by a limiting process either from Problem 3 with $\mu \to 0$, from Problem 4 with $\alpha \to 0$, or from Problem 5 with $\beta \to 0$. The results are

$$y_H(x) = A_1 + A_2 x \tag{34}$$

$$
\begin{aligned}
G_1(x) &= (M_0/2EI)z_1^2 \qquad & G_2(x) &= (W/6EI)z_2^3 \\
G_3(x) &= (w/24EI)z_3^4 \qquad & G_4(x) &= (w_0/120EIa_4)z_4^5
\end{aligned}
\tag{35}
$$

Problem 7. A Uniform Beam Vibrating Steadily Under the Action of Pulsating Discontinuous Loads

Consider a uniform beam that is subjected to a set of pulsating lateral loads, each being proportional to $\sin\Omega t$. Let the deflection of the beam, when steady conditions prevail, be $y(x)\sin\Omega t$. Then, with the usual neglect of the angular inertia of elements of the beam, the governing differential equation is

$$EIy'''' - m\Omega^2 y = p(x) \tag{36}$$

where $p(x)$ is the amplitude of the applied lateral load per unit length and m is the mass per unit length.

Consider the effect of the four types of discontinuous load shown in Fig. 1, where now M_0, W, w, and w_0 represent the amplitudes of the four loads. The right-hand side of Eq. (36) is still given by Eq. (10) and the required continuity conditions at the points $x = a_1, a_2, a_3, a_4$ are identical with Eqs. (11).

Letting

$$\gamma = (m\Omega^2/EI)^{1/4} \tag{37}$$

the solution is as follows:

$$y_H = A_1\sin\gamma x + A_2\cos\gamma x + A_3\sinh\gamma x + A_4\cosh\gamma x \tag{38}$$

$$
\left.
\begin{aligned}
G_1(x) &= (M_0/2EI\gamma^2)(\cosh\gamma z_1 - \cos\gamma z_1) \\
G_2(x) &= (W/2EI\gamma^3)(\sinh\gamma z_2 - \sin\gamma z_2) \\
G_3(x) &= (w/2m\Omega^2)(\cosh\gamma z_3 + \cos\gamma z_3 - 2) \\
G_4(x) &= (w_0/2\gamma m\Omega^2 a_4)(\sinh\gamma z_4 + \sin\gamma z_4 - 2\gamma z_4)
\end{aligned}
\right\}
\tag{39}
$$

If $\Omega \to 0$ so that $\gamma \to 0$, Eqs. (39) become in the limit identical with Eq. (35).

Some Applications to Plate Problems

Problem 8. A Circular Plate Subjected to Axially Symmetrical Discontinuous Lateral Loads

Consider a thin horizontal circular plate of thickness t, Young's modulus E, and Poisson's ratio ν, which is bent axially symmetrically by a set of lateral loads. According to small deflection theory, the lateral deflection y at radius r satisfies the differential equation

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dy}{dr}\right)\right] = \frac{d^3y}{dr^3} + \frac{1}{r}\frac{d^2y}{dr^2} - \frac{1}{r^2}\frac{dy}{dr} = \frac{F(r)}{D} \tag{40}$$

where $F(r)$ is the vertical shear force per unit of circumference due to the lateral loads, and D is the flexural rigidity of the plate defined by the equation

$$D = Et^3/12(1 - \nu^2)$$

Consider the effect of the three types of discontinuous loading shown in Fig. 2, namely, a radial bending moment M_0 per unit circumference distributed uniformly around a circle of radius a_1, a line load of magnitude W per unit circumference distributed uniformly around a circle of radius a_2, and a uniform pressure p applied over the annular region $a_3 \leqq r \leqq b$, where b is the outside radius of the plate.

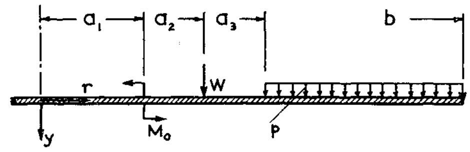

Fig. 2 Diametral section of circular plate showing types of discontinuous loading.

Because of these loads alone, the normal shear force $F(r)$ is given by

$$F(r) = W\langle a_2/r\rangle_{a_2} + \tfrac{1}{2}p\langle r - (a_3^2/r)\rangle_{a_3} \tag{41}$$

Since Eq. (40) is of third order, three continuity conditions are required at each of the points $x = a_1, a_2, a_3$. These are

$$
\begin{aligned}
\Delta y(a_i) &= \Delta y'(a_i) = 0 \qquad & i &= 1, 2, 3 \\
\Delta y''(a_1) &= M_0/D \qquad & \Delta y''(a_2) &= \Delta y''(a_3) = 0
\end{aligned}
\tag{42}
$$

The solution $y_H(r)$ and the deflection functions $G_i(r)$ which appear in the Macaulay brackets in the equation for $y(r)$ are listed below:

$$y_H(r) = A_1 r^2 + A_2\log r + A_3 \tag{43}$$

$$
\begin{aligned}
G_1(r) &= (M_0/4D)[r^2 - a_1^2 - 2a_1^2\log(r/a_1)] \\
G_2(r) &= (Wa_2/4D)[(r^2 + a_2^2)\log(r/a_2) - (r^2 - a_2^2)] \\
G_3(r) &= (P/64D)[(r^2 - a_3^2)(r^2 + 5a_3^2) - \\
&\qquad 4a_3^2(2r^2 + a_3^2)\log(r/a_3)]
\end{aligned}
\tag{44}
$$

Problem 9. A Uniformly Compressed Circular Plate Subjected to Axially Symmetrical Discontinuous Lateral Loads

Suppose now that, in addition to the lateral loads, the plate of Problem 8 is subjected to a uniform membrane compression of magnitude P per unit length in all directions. This would occur, for example, if the plate did not contain a hole and if it were subjected to a uniform radial compressive force P per unit length at its outer circumference.

The differential equation now becomes

$$\frac{d^3y}{dr^3} + \frac{1}{r}\frac{d^2y}{dr^2} + \left(\kappa^2 - \frac{1}{r^2}\right)\frac{dy}{dr} = \frac{F(r)}{D} \tag{45}$$

where

$$\kappa = (P/D)^{1/2} \tag{46}$$

Equations (41) and (42) are still applicable and the solution is as follows:

$$y_H(r) = A_1 J_0(\kappa r) + A_2 Y_0(\kappa r) + A_3 \tag{47}$$

$$
\left.
\begin{aligned}
G_1(r) &= (M_0a_1/\kappa D)[(1/\kappa a_1) + (\pi/2)\{Y_1(\kappa a_1)J_0(\kappa r) - \\
&\qquad J_1(\kappa a_1)Y_0(\kappa r)\}] \\
G_2(r) &= (Wa_2/P)[\log(r/a_2) + (\pi/2)\{Y_0(\kappa a_2)J_0(\kappa r) - \\
&\qquad J_0(\kappa a_2)Y_0(\kappa r)\}] \\
G_3(r) &= (pa_3/\kappa P)[\tfrac{1}{4}\kappa a_3\{(r^2/a_3^2) - 1 - 2\log(r/a_3)\} - \\
&\qquad (1/\kappa a_3) - (\pi/2)\{Y_1(\kappa a_3)J_0(\kappa r) - J_1(\kappa a_3)Y_0(\kappa r)\}]
\end{aligned}
\right\}
\tag{48}
$$

In these expressions J_0, Y_0 and J_1, Y_1 are Bessel functions of the first and second kinds, of zero and unit order, respectively. In deriving Eqs. (48), use has been made of the following identities:

$$
\begin{aligned}
J_0'(z) &= -J_1(z) \qquad Y_0'(z) = -Y_1(z) \\
J_1(z)Y_0(z) - J_0(z)Y_1(z) &= J_1(z)Y_1'(z) - \\
&\qquad Y_1(z)J_1'(z) = (2/\pi z) \\
J_1'(z)Y_0(z) - J_0(z)Y_1'(z) &= -(2/\pi z^2)
\end{aligned}
\tag{49}
$$

Problem 10. A Uniformly Stretched Circular Plate Subjected to Axially Symmetrical Discontinuous Lateral Loads

This is identical with Problem 9 except that the plate is now subjected to a uniform membrane tension of magnitude T per unit length in all directions, instead of the compression P.

The differential equation is the same as Eq. (45) except that κ^2 is replaced by $-\tau^2$ where

$$\tau = (T/D)^{1/2} \tag{50}$$

and the solution is given by

$$y_H(r) = A_1 I_0\langle\tau r) + A_2 K_0\langle\tau r) + A_3 \tag{51}$$

$$\left.\begin{aligned} G_1(r) &= (M_0a_1/\tau D)[K_1(\tau a_1)I_0\langle\tau r) + I_1(\tau a_1)K_0(\tau r) - (1/\tau a_1)] \\ G_2(r) &= (Wa_2/T)[K_0(\tau a_2)I_0(\tau r) - I_0(\tau a_2)K_0\langle\tau r) - \log(r/a_2)] \\ G_3(r) &= (pa_3/\tau T)[K_1(\tau a_3)I_0(\tau r) + I_1(\tau a_3)K_0(\tau r) - (1/\tau a_3) + \tfrac{1}{2}\tau a_3\{1 - (r^2/a_3^2) + 2\log(r/a_3)\}] \end{aligned}\right\} \tag{52}$$

where I_0, K_0 and I_1, K_1 are modified Bessel functions of the first and second kinds, of zero and unit order, respectively. The following identities have been used in deriving Eqs. (52):

$$I_0'(z) = I_1(z) \qquad K_0'(z) = -K_1(z)$$

$$\begin{aligned} K_1(z)I_0(z) + I_1(z)K_0(z) &= K_1(z)I_1'(z) - I_1(z)K_1'(z) = 1/z \\ I_1'(z)K_0(z) + K_1'(z)I_0(z) &= -1/z^2 \end{aligned} \tag{53}$$

It may be noted that the heaviness of the algebra in a paper by Hicks,[9] which derives expressions for the bending moments in uniformly stretched plates due to loads similar to W and p in Fig. 2, could have been reduced considerably by the introduction of appropriate Macaulay brackets.

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Influence of Tesseral Harmonics on Nearly Circular Polar and Equatorial Orbits

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Approximate closed form expressions are obtained for the radial, cross-track, in-track, nodal period, and sidereal period perturbations due to the second sectorial ($J_2^{(2)}$) harmonic. The effects of higher-order tesseral harmonics on the position coordinates are analyzed by means of special perturbation programs. For low-altitude orbits, radial perturbations as large as 1200 ft occur due to $J_2^{(2)}$. Secular in-track perturbations (with respect to an unperturbed orbit) due to $J_2^{(2)}$ are found to be as large as 0.012°/rev, whereas cross-track perturbations are periodic with maximum amplitudes of 0.02°. Periodic oscillations occur in the orbital period with amplitudes of ±0.003 min about a mean value that differs from the unperturbed value. Of the higher-order tesserals that are investigated, the $J_3^{(3)}$ term is most influential, contributing perturbations of the same order as $J_2^{(2)}$.

Nomenclature

r = radial distance to the satellite from the center of the earth
g_r = radial component of acceleration due to the tesseral harmonics in the earth's gravity potential
g_θ = component of acceleration in the θ direction due to the tesseral harmonics
g_ϕ = component of acceleration in the ϕ direction due to the tesseral harmonics
θ = position coordinate measured in the longitudinal direction
ϕ = position coordinate measured in the direction of increasing latitude
μ = universal gravitational constant times mass of the earth
r_c = radius of reference circular orbit (const)
r_1 = perturbation in the radius due to g_r, g_θ, and g_ϕ, and the departure from circularity in the initial conditions
θ_1 = perturbation in the θ direction due to g_r, g_θ, and g_ϕ, and initial conditions
ϕ_1 = perturbation in the ϕ direction due to g_r, g_θ, and g_ϕ, and initial conditions
ϕ_c = angular position (measured from the equator in the direction of increasing latitude) that the satellite would have in the reference circular orbit; $\phi_c = \dot\phi_c t$
θ_c = angular position (measured in the direction of increasing longitude) that the satellite would have in the reference circular orbit; $\theta_c = \dot\theta_c t$
$(\dot{\ })$ = derivative with respect to time

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